A Pedagogical Model of CO_2 and O_2 Atmospheric Abundances and Tree Population due to Human Population

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Abstract. A pedagogical model of the effects of human population on the global tree population and the atmospheric abundances of carbon dioxide and oxygen is provided, which, though too simple to be precise, offers meaningful insights with the virtue of being solvable by analytical means using only elementary calculus.

Introduction

This paper presents a model ecosystem of human and tree populations living in an atmosphere of carbon dioxide and oxygen. It addresses the effects of a growing human population and a declining tree population on the CO_2 and O_2 abundances. Full disclosure requires acknowledging the model's oversimplified nature. It is to climatology what frictionless ramps and circular planetary orbits are to mechanics—idealized models intended to illustrate strategy over precision. It provides a pedagogical stepping-stone toward more realistic models.

The dynamics of greenhouse gases and their effects on climate are well studied.[1] Thus our simple model offers no new climatology results. Rather, it offers experience in thinking about issues that arise in such models and can be solved analytically using only introductory calculus.

Model

Let t denote time, with $t = 0$ in the year 1500, because we are interested in the industrial era that followed. Our calculations were done in 2018, or $t = 518$ yr. Let $P(t)$ denote the human population in number of individuals; $C(t)$ and $O(t)$ the atmospheric oxygen abundance in tons, respectively. For boundary conditions we borrow data from several authors: $P(0) \equiv P_o$ = world human population in 1500 = 0.4×10^9;[2] global population in 2018 = P(518 yr) = 7.6×10^9 persons;[2] from Wolchover,[3] $P(\infty) \equiv P_\infty$ = human population at global saturation (carrying capacity) = 20×10^9;[3] from Amos[4] and Bolton,[5] T_o = tree population in 1500 = 6×10^{12}, and T(518 yr) = 3×10^{12} trees. Amos's data[4] places the present tree loss rate between 10 billion and 15 billion trees annually[4]:

$$\left[\frac{dT}{dt}\right]_{2018} = -15\{10\} \times 10^9 \text{ trees/yr}$$

In our model we take the 15 billion value, but in calculations that depend on this rate we include in brackets { } for comparison results for the 10 billion annual loss. For the reader's reference, initial conditions and other relevant constants are gathered in Table 1.

Turning to the rate equations, the human population growth rate is proportional to the current population and to the difference between the present and saturation levels; hence

$$\frac{dP}{dt} = P\lambda\left(1 - \frac{P}{P_\infty}\right) \tag{1}$$

with rate coefficient λ.

The tree population declines due to human-caused deforestation at a rate proportional to P. For several decades the human deforestation rate has been on the order of an acre per second.[6] Since human deforestation dominates over natural tree death, for the tree population rate equation we write

$$\frac{dT}{dt} = -(\mu P)T \tag{2}$$

with rate coefficient μ.

TABLE 1. Rate constants used in the model. Numbers in brackets { } use the upper estimate on current annual tree loss.

Initial human population:	$P_o = 0.4 \times 10^9$ persons
Carrying capacity:	$P_\infty = 20 \times 10^9$ persons
Fraction of carrying capacity at $t = 0$:	$\rho = 0.02$
Initial tree population:	$T_o = 6 \times 10^{12}$ trees
2018 tree population:	$T(518) = 3 \times 10^{12}$ trees
2018 tree loss rate:	$[dT/dt]_{2018} = 10 - \{15\} \times 10^9$ trees/yr
Human population rate constant:	$\lambda = 0.0109/$ yr
Human-caused tree loss rate constant:	$\mu = 1.06 \times 10^{-13}$/person-yr
Rate const., O_2 production by trees:	$k_1 = 0.13$ T/tree-yr
Rate const., O_2 consumption by people:	$k_2 = 1.8 \times 10^{-4}$ T/person-yr (breathing only)
	12.5 T/person-yr (including machines)
Rate const., CO_2 people & machines:	$k_3 = 11$ T/person-yr
CO_2 absorbed per tree per year	$k_4 = 0.024$ tons/tree-yr
$\gamma \equiv \mu P_\infty/\lambda$	$\gamma = 1.2$ {0.8}

Humans and their machines produce carbon dioxide with rate coefficient k_3, while trees consume carbon dioxide with rate coefficient k_4. Neglecting other sinks of carbon dioxide (such as the ocean) because we restrict our study to the effects of trees only, we write

$$\frac{dC}{dt} = k_3 P - k_4 T \tag{3}$$

Oxygen is produced by trees with rate coefficient k_1 and is consumed by people and their machines with rate coefficient k_2; hence

$$\frac{dO}{dt} = k_1 T - k_2 P \tag{4}$$

Eqs. (1)–(4) are schematically represented in Fig. 1.

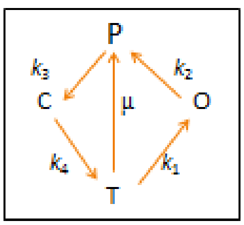

FIG. 1. CO_2 and O_2 flow diagram described by our model's rate equations, which includes human population P, tree population T, atmospheric carbon dioxide abundance C, and oxygen abundance O. Rate coefficients are denoted on the directed lines.

In general the rate coefficients are not constants. For instance, in 1967 the human population growth rate coefficient λ exceeded 2%/yr, but by 2018 it dropped to 1.09%/yr. Thus for maximum realism one should solve Eqs. (1)–(4) numerically, stepping through time slices of duration Δt with elapsed time $t_n = n\Delta t$ (n is a non-negative integer). For example, Eq. (1) would become

$$P(t_{n+1}) = P(t_n) + (\Delta t)\lambda(t_n)P(t_n)\left[1 - \frac{P(t_n)}{P_\infty}\right] \tag{5}$$

However, because we seek an illustrative analytic solution, we approximate all the rate coefficients as constants. For their values we borrow published data: The human population growth rate in 2018 is $\lambda = 1.09\%$/yr;[2] the tree loss rate coefficient μ can be estimated from Eq. (2) using contemporary values of Amos[4]:

$$\mu = \left[\frac{1}{TP}\frac{dT}{dt}\right]_{2018} = 1.06 \times 10^{-13}/\text{person} - \text{yr}$$

From Ref. [7] we obtain k_1 = oxygen production rate per tree per year = 260 lb/tree-yr = 0.13 tons/tree-yr; and from Ref. [8], k_2 = oxygen consumed per person (breathing only) per year = 0.17 kg/person-yr = 1.8×10^{-4} tons/person-yr (the oxygen consumption of machines will be addressed later); k_3 = annual CO_2 production (in 2018) by humans and their machines = 11 tons/person-yr;[9] from New York State University data,[10] k_4 = CO_2 absorbed by one tree per year

= 48 lb/tree-yr = 0.024 tons/tree-yr.[10] The coefficients k_1 and k_3 are determined by biology and are essentially constant on human timescales. In contrast, k_2 and k_3 depend on—and are dominated by—technology.

Consider what must happen among the rate coefficients in order to have equilibrium between the CO_2 and O_2 abundances. Setting their rates of change equal to zero, Eqs. (3) and (4) require the k_3 equilibrium value k_{3o} to satisfy

$$k_{3_o} = \frac{k_2 k_4}{k_1} = 0.067\,\mathrm{lb/person-yr} \tag{6}$$

But in 2018, k_3 = 11 tons/person-yr, greater than our equilibrium value by a factor of 328,400. This suggests that humanity's relationship with the atmosphere may be unsustainable.

For breathing only, each person needs about 1.8 × 10^{-4} tons of oxygen per year.[8] Each tree produces about 0.13 tons/yr of oxygen.[7] Thus each tree can supply N persons with just enough oxygen for breathing, where

$$Nk_2 = k_1 \tag{7}$$

which with our assumed value of k_1 and k_2 gives $N \approx 716$—one tree can supply 716 people with just enough oxygen necessary for life.[11] The 2018 tree population is about 3 trillion,[4,5] which means that with these numbers, the present tree population by itself can support about 4.4 billion people—about half the current population. Clearly our model is too simple; for example, we neglect photosynthesis of other plants and ocean phytoplankton, and these estimates do not include oxygen consumed in burning fossil fuels. But within the world described by our model, each tree can support 716 people, so in this model a critical time t_c in the human–ecosystem relationship occurs when

$$\frac{T(t_c)}{P(t_c)} = 716 \tag{8}$$

We next derive expressions for $T(t)$ and $P(t)$.

Solutions to Rate Equations

Equation (1) can be integrated by separation of variables, which gives

$$P(t) = \frac{P_o P_\infty e^{\lambda t}}{P_\infty - P_o + P_o e^{\lambda t}} \tag{9}$$

If $P_o \ll P_\infty$, then

$$P(t) \approx \frac{P_o e^{\lambda t}}{1+\rho e^{\lambda t}} \tag{10}$$

where $\rho \equiv P_o/P_\infty$ denotes the fraction of population capacity reached in the year 1500. With our numbers, $\rho = 0.02$.

Using Eq. (10) in Eq. (2) allows another separation of variables, yielding

$$T(t) = T_o \left(\frac{1+\rho e^{\lambda t}}{1+\rho}\right)^{-\gamma} \tag{11}$$

where $\gamma \equiv \mu P_\infty/\lambda = 1.2$ {0.8}. Since $\rho << 1$ then

$$T(t) \approx T_o\left(1+\rho e^{\lambda t}\right)^{-\gamma} \tag{12}$$

For sufficiently large times, when $\rho e^{\lambda t} >> 1$, even though ρ << 1, Eqs. (14)–(12) become $T(t) \approx T_o \rho e^{-\gamma\lambda t}$, an exponential decline in the tree population with half-life $t_{1/2} = \frac{\ln 2}{\gamma\lambda}$ = 53 yr {79 yr}. To examine the early behavior of $T(t)$, expand the right-hand side of Eq. (12) in a Taylor series about $t = 0$ and approximate $1+\rho \approx 1$. These steps result in

$$T(t) \approx T_o\left(1 - \rho\gamma\lambda t + \frac{1}{2}\rho\gamma^2\lambda^2 t^2 + \cdots\right) \tag{13}$$

Since $\gamma\lambda = \mu P_\infty$, Eq. (13) can alternatively be written

$$T \approx T_o\left(1 - \rho\mu P_\infty t + \frac{1}{2}\rho(\mu P_\infty)^2 t^2 + \cdots\right) \tag{14}$$

where $\mu P_\infty \sim 10^{-3}$/yr and $\rho = 0.02$. Equation (13) shows a linear decline in trees at times shortly after the year 1500. When did nonlinearity in Eq. (14) become apparent? Compare Eq. (13) to the Taylor series expansion of a function $f(t)$ to second order, where $f(0) = 1$. The quadratic term becomes apparent when

$$\frac{t^2}{2} f''(0) = \alpha \tag{15}$$

where α is just large enough to be detectable. If $f(t) = T(t)/T_o$, then Eq. (15) gives

$$t = \frac{1}{\mu P_\infty}\sqrt{\frac{2\alpha}{\rho}} \approx \sqrt{\alpha} \times 10^4\ \mathrm{yr} \tag{16}$$

If nonlinearity is detectable when α is, say, one-tenth of one percent, then in our model t = 316 yr, the year 1816, and we are now well into a nonlinear decline of the tree population.

Let us return to the critical time defined by Eq. (8), when the number of trees per person equals the minimum necessary to support human life (not to mention the lives of other oxygen-breathing species). Let n be the number of persons per tree when $t = t_c$, which with our numbers of 716 trees/person gives $n = 0.0013$ persons/tree. Inserting Eqs. (10) and (14) into Eq. (8) gives

$$\frac{nT_o}{P_o} = x(1+\rho x)^{\gamma-1} \qquad (17)$$

where $nT_o/P_o = 1.07 \times 10^7$ and $x = e^{\lambda t_c} \equiv 10^{7+\varepsilon}$. A numerical solution of Eq. (17) shows $\varepsilon = -0.856$ {+1.44}, so that $e^{\lambda t_c} = 1.4 \times 10^6$ {2.75×10^8}, and $t_c = 1298$ {2001}, the year 2798 {3501}. (In 2018 there were about 7 billion people and 3 trillion trees, so $n_{2018} \sim 0.002$ persons/tree.)

Next we turn to the rate equations for carbon dioxide and oxygen. Using Eqs. (10) and (12), upon integration Eq. (3) becomes

$$C(t) - C_o = k_3 P_o\, I(t) - k_4 T_o\, J(t) \qquad (18)$$

and for Eq. (4),

$$O(t) - O_o = -k_2 P_o\, I(t) + k_1 T_o\, J(t) \qquad (19)$$

where C_o and O_o are integration constants,[12] with

$$I(t) = \int_0^t \frac{e^{\lambda t'}}{1+\rho e^{\lambda t'}}\, dt' = \frac{1}{\rho\lambda}\ln\left(1+\rho e^{\lambda t}\right) \qquad (20)$$

and

$$J(t) = \int_0^t (1+\rho e^{\lambda t'})^{-\gamma}\, dt' \qquad (21)$$

One can try several approximation schemes. For instance, at sufficiently large times when $\rho e^{\lambda t} >> 1$,[13] we may say $I(t) \approx \frac{1}{\rho\lambda}(\ln\rho + \lambda t)$, and $J(t)$ becomes

$$J(t) \approx \frac{1}{\rho^\gamma \gamma\lambda}\left(1 - e^{-\gamma\lambda t}\right) \qquad (22)$$

Now Eq. (18) becomes approximately

$$C(t) \approx C_o + \frac{k_3 P_o}{\rho\lambda}(\ln\rho + \lambda t) - \frac{k_4 T_o}{\rho^\gamma\gamma\lambda}\left(1 - e^{-\gamma\lambda t}\right) \qquad (23)$$

and Eq. (19),

$$O(t) \approx O_o + \frac{k_2 P_o}{\rho\lambda}(\ln\rho + \lambda t) + \frac{k_1 T_o}{\rho^\gamma\gamma\lambda}\left(1 - e^{-\gamma\lambda t}\right) \qquad (24)$$

It is revealing to separate the two contributions to the *changes* in the CO_2 and oxygen abundances from 1500 to 2018. Using our data in Eq. (23), the net change in CO_2 shows an increase of over 8 GT:

$$C(518\text{ yr}) - C_o \approx (81.4 - 0.1) \times 10^{12}\text{ tons} = 81.3\text{ GT} \qquad (25)$$

where 81.4 GT comes from people with their machines producing 11 T/person-yr of CO_2, and the 0.1 GT comes from CO_2 uptake by trees. The oxygen difference, using $k_2 = 1.8 \times 10^{-4}$ tons/person-yr for *breathing only*, gives a result that, when compared to the effects of technology, is quite revealing. Then the terms in Eq. (24) show a net O_2 decrease on the order of 0.8 GT:

$$O(518\text{ yr}) - O_o \approx (-1.34 + 0.54)\text{ GT} = -0.80\text{ GT} \qquad (26)$$

The negative balance means that when people use oxygen only for breathing, oxygen consumption due to population growth depletes the oxygen supply faster than trees can replenish it. Figure 2 shows $C(t) - C_o$ and $O(t) - O_o$ as functions of time. Notice that the CO_2 abundance is essentially flat until the year 1900, the O_2 abundance declines sharply almost coincidentally with the CO_2 increase, and these quantities become equal at approximately $t = 600$ yr, the year 2100. After that CO_2 increases and O_2 decreases approximately linearly, according to Eqs. (23)–(24).

However, each person with their machines produce about 11 tons of CO_2 per year (2018 figures).[9] Of this, only 1.8×10^{-4} tons are exhaled in breathing,[9] and therefore essentially all of the 11 tons/yr is produced by machines. How much annual per capita O_2 consumption does this imply? Consider the combustion of octane, the dominant molecule in gasoline. Its combustion proceeds according to the reaction

$$C_8H_{18} + 12.5\, O_2 \rightarrow 8CO_2 + 9H_2 0 \qquad (27)$$

The weight ratio of eight CO_2 molecules to one octane molecule is $[8CO_2]/[C_8H_{18}] \approx 3$. A gallon of gasoline weighs approximately 6 pounds. Therefore the combustion of one gallon of gasoline produces about 18 pounds of CO_2. The weight ratio of 12.5 O_2 molecules to 8 CO_2 molecules is $[12.5\, O_2]/[8\, CO_2] \approx 1.136$, so to produce 11 tons of CO_2 consumes about 12.5 tons of O_2 per capita each year. Taking into account the oxygen consumption by machines per capita, the rate coefficient for O_2 consumption changes k_2 into 12.5 tons/person-yr, which in turn changes Eq. (26) into

$$O(518\text{ yr}) - O_o \approx -92\text{ GT} \qquad (28)$$

The oxygen consumption of machines accounts for an "excess" oxygen consumption of over 90 billion tons.

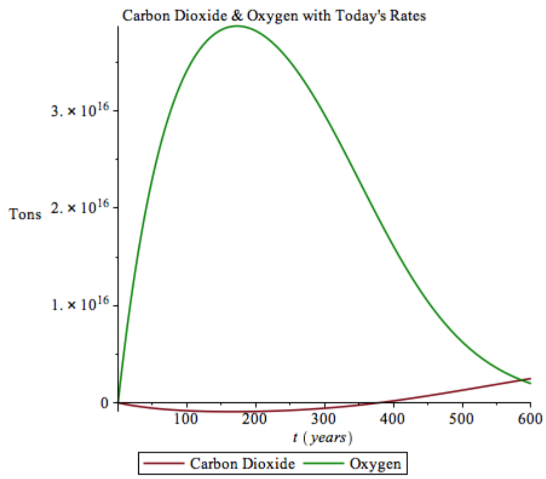

FIG. 2. The model's atmospheric carbon dioxide and oxygen abundances, $C(t) - C_o$ and $O(t) - O_o$, using rate coefficients held at their 2018 values. $t = 0$ denotes the year 1500.

Discussion

The foregoing calculations are a toy model intended to demonstrate the issues involved in studying the dynamics of atmospheric carbon dioxide and oxygen abundances as they are affected by trees and people *only*. The final numbers produced in this model are not meant to be taken seriously, but they do suggest qualitative trends that we as a society, and as individuals, would do well to take seriously. As Freeman Dyson has observed, "In the long run, qualitative changes always outweigh the quantitative ones."[14]

We have chosen the year 1500 as $t = 0$ because it was near the end of the pre-industrial era, before fossil fuel burning became the norm and before much of the planet was deforested, and because robust estimates exist of the human and tree populations at that time.

Most of the rate coefficients are not constants—a feature we have ignored for mathematical simplicity. To step through Eqs. (1)–(4) numerically would require data for the rate coefficients as a function of time, a task beyond the scope of this study. More realistic models would also include other sources and sinks of carbon dioxide and oxygen (e.g., the ocean's phytoplankton produces at least half of the oxygen[15]), other agents besides human actions that affect tree population, and other atmospheric gases.

One point is certain: This study reinforces the realization that unabated fossil fuel consumption, deforestation, and exponential population growth are not sustainable. Human economies and desires are not immune to nature's realities.

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11. Reference 4 mentions that in 2015 there were “upwards of 420 trees for every person on the planet.”
12. From Eq. (23) C_o may be written $C_o = C(0) - (k_3 P_o/\rho\lambda) \ln \rho$ and similarly for O_o.
13. The approximation described in Eqs. (23) and (24) will be quite inaccurate for decades shortly after 1500. An alternative approximation comes from noting that the two values of γ average to 1, so setting $\gamma = 1$ offers another approach with different details but similar qualitative results compared to Eqs. (23) and (24). The purpose of this exercise is not “the answer” but glimpsing the strategic decisions that must be made in modeling complex phenomena.
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