Elegant Connections in Physics

continued from

Strange Consequences of the Finite Speed of Light

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Retarded Time, Jefimenko's Equations, and the Liénard-Wiechert Potentials

From the perspective of classical electrodynamics, light is a wave in the electromagnetic field. However, our studies of electromagnetism usually begin with electrostatics and magnetostatics. For instance, according to Coulomb's law the electrostatic field **E** due to a point charge q is $\mathbf{E} \sim q\mathbf{R}/R^3$, where **R** denotes the displacement vector from the charge to the observer. Moving beyond electrostatics, if the charge q somehow became time-dependent, so that $q = q(t)$, then as q changes, **E** changes. The change in q produces a ripple in **E** that propagates at the speed of light. Thus the field detected by the observer at time t would depend on the value of q at the time R/c earlier. Therefore, shouldn't the time-dependent **E** simply be $\mathbf{E}(t) \sim q(t - R/c)\mathbf{R}/R^3$, where **R** means the vector from the charge's location when it emitted the change in the field whose ripple arrived at the observer's location at the later time t? If that were so, then a time-dependent electric field could be envisioned as a time-ordered sequence of electrostatic Coulomb fields. And shouldn't the same adjustment for time-dependent currents be made to take us from the Biot-Savart law of magnetostatics to time-dependent magnetic fields? If these scenarios were all that was necessary in going from static fields to electrodynamics, electromagnetism might be a lot simpler but it would not be nearly so interesting. It would also be wrong. It turns out that this simple strategy works for the electric and magnetic *potentials*, but not for **E** and **B** themselves.

Maxwell's equations are four coupled first-order partial differential equations for the electric field **E** and the magnetic field **B**. They relate the electromagnetic field's sources—charge density ρ and current density **J**—to the fields, and the fields to each other. The fields **E** and **B** are derivatives of the potentials φ and **A**:

$$E = -\nabla\varphi - \frac{\partial A}{\partial t} \qquad (3)$$

and

$$B = \nabla \times A. \qquad (4)$$

In electrostatics the potential is given by

$$\varphi(r) = k\int \frac{\rho(r')}{|r-r'|}\, d^3r' \qquad (5)$$

where $\mathbf{r} - \mathbf{r'} \equiv \mathbf{R}$ denotes the vector from the source point **r'** to the field point **r**, d^3r' a volume element for integrating over source coordinates, $k = 1/4\pi\varepsilon_o$ denotes the Coulomb constant, the integral extends over all space, and the charge density vanishes at infinity rapidly enough for the integral to exist. In magnetostatics the vector potential is

$$A(r) = k_m\int \frac{J(r')}{|r-r'|}\, d^3r' \qquad (6)$$

where $k_m = \mu_o/4\pi$ denotes the Biot-Savart constant. When the sources and thus the fields are allowed to be time-dependent, the charge and current densities become functions of time, $\rho = \rho(\mathbf{r'},t)$ and $\mathbf{J} = \mathbf{J}(\mathbf{r'},t)$. However, it

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takes the time $|\mathbf{r} - \mathbf{r}'|/c$ for the change in the field to propagate from a source point to the field point. One way to solve a problem is to guess the solution, so let us guess that the time-dependent potentials take on the simplest possible upgrade (the same strategy proposed above for transitioning from static **E** and **B** to their time-dependent extensions, but here applied to the potentials): keep the spatial structure of Eqs. (5) and (6) intact, but require that the potentials at the field point, at time t, depend on what the sources at **r'** were doing at the time R/c earlier. This intuitive argument motivates one to write

$$\varphi(r,t) = k\int \frac{\rho(r',t_R)}{|r-r'|}\, d^3r' \qquad (7)$$

and

$$A(r,t) = k_m\int \frac{J(r',t_R)}{|r-r'|}\, d^3r' \qquad (8)$$

which introduces the so-called "retarded time" t_R,

$$t_R \equiv t - R/c. \qquad (9)$$

When the intuitive solution thus assumed for the potentials, if it *is* correct it can be made rigorous after the fact by showing that it satisfies the original inhomogeneous wave equation (in the Lorenz gauge). The guess turns out to be correct, so as you would expect it can also be derived deductively (no guessing!) with rigor.[6] The fields **E** and **B** follow from applying Eqs. (1) and (2) to Eqs. (7) and (8), a somewhat laborious task because the gradient and curl pick up ∇t_R terms. *These terms would have been missed had we made the substitution* $t \rightarrow t_R$ *in the static* **E** and **B** *fields directly*. After taking the derivatives, at the end of the day one obtains **E** and **B** as integrals over the source densities, Jefimenko's equations:[7]

$$E(r,t) = k\int\left[\frac{\rho\hat{R}}{R^2} + \frac{\hat{R}}{Rc}\frac{d\rho}{dt} - \frac{1}{Rc^2}\frac{dJ}{dt}\right]d^3r' \qquad (10)$$

and

$$B(r,t) = k_m\int\left[\frac{J\times\hat{R}}{R^2} + \frac{1}{Rc}\left(\frac{dJ}{dt}\right)\times\hat{R}\right] d^3r'. \qquad (11)$$

Jefimenko's equations are gauge-invariant, and reveals the interesting subtly of electrodynamics that forms the focus of our attention. The correct time-dependent *potentials* could be guessed by replacing static charge and current densities with time-dependent ones, with allowance made for the retarded time. However, had that strategy been applied to Coulomb's law for **E** and the Biot-Savart law for **B**, the $1/R$ electromagnetic field that accounts for radiation would have been missed entirely, because the fun with the ∇t_R terms would have been bypassed. There is a lot of physics in the spatial variation of the retarded time.

In the term $R/c \equiv |\mathbf{r} - \mathbf{r}'|/c$ of Eq. (9), and in its applications in Eqs. (7)-(8) and (10)-(11), **r'** is where a source particle was *when the signal was emitted at time* t_R, and **r** is where the observer detected the signal at the later time t. Conceptually, this seems straightforward enough. However, when sitting down to solve a problem one can feel as ifhaving been placed between parallel mirrors. For instance, consider a source moving with constant velocity **v**, such that $\mathbf{r}' = \mathbf{v}t$. When I replace t with t_R, do I write $t_R = t - |\mathbf{r} - \mathbf{v}t|/c$ and let it go at that, or do I write $t - |\mathbf{r} - \mathbf{v}(t - R/c)|/c$ and continue iterating? The point to remember in retarded time problems is that, no matter how crazy the motion of a point source q, at any one instant t the observer can receive a signal from only one past event in q's history. This is so because a charged particle cannot move at the speed of light relative to *any* observer. Thus if q sends forth a signal from event A, the signal will always outrace q itself, so I will detect, at any time, at most one signal that was emitted by q.[8] Thus when I want to evaluate integrals such as Eq. (7) or (8), the question becomes: what time was it (according tothe synchronized clocks in my reference frame that measure events *locally* and then report their data to me later when I analyze the experiment) *when the source charge sent forth that signal that I receive at the instant t*? Let us call the time of emission t' (which is the same as t_R but the new notation emphasizes that t' was the *moment*, according to my reference frame's clocks, when the signal was emitted from **r'**, whereas the symbol t_R emphasizes the intervening *distance R*). If I know the source particle's trajectory **r′** as a function of time, then t', the time the signal was emitted that later landed in my camera t, can be found in terms of **r** and t by solving for t' in Eq. (9), written as

$$c(t - t') = |\mathbf{r} - \mathbf{r}'(t')|. \qquad (12)$$

One solves this equation to obtain an expression $t' = f(t,\mathbf{r})$ that gives the instant t' when the signal emitted by q arrives at my detector at time $t > t'$. In the integrals of Eqs. (7) and (8), we integrate over the **r'** *spatial* coordinates; Eq. (12) tells us the time t', and thus the location $\mathbf{r}'(t')$, when a charge emitted its signal that landed at **r** at time t.

The aforementioned confusion gets cleared up when separating the applications of Eqs. (7) and (8) into two kinds of situations, (A) charge and current distributions that vary their strength in time but don't move around, and (B) moving point charges. For scenario (A), the charges and currents stay in place but their amounts vary with time. Examples include a capacitor being charged, where fields are shielded (or ignored) that arise from the current delivering charge to the capacitor. As a subset of such problems, when the size of the source charge and current arrays are small compared to the distance between them and the field point, so that $|\mathbf{r}'| << |\mathbf{r}|$ for all **r'**, an expansion of $1/|\mathbf{r} - \mathbf{r}'|$ about $\mathbf{r}' = \mathbf{0}$ turns Eqs. (7) and (8) into multipole expansions. Applications include the radiation patterns produced by the electric and magnetic dipole antennas.[9] Case (B) considers the radiation produced by a single point charge moving along some trajectory **r′**, known as a function of time. Once we know how to find the **E** and **B** fields for an arbitrarily moving single point charge, the fields detected at **r** at time t, produced by arrays of moving charges, follow by superposition.

In situation (A) bodies carrying charge q and wires carrying currents I stay at rest, but the charges on those bodies and the currents in those wires vary with time, $q = q(t)$ and $I = I(t)$. Consider the problem of determining the fields on the axis (the z axis) of a uniformly charged disc of radius a. In the electrostatics case, an introductory physics example has the student show from Eq. (5) that

$$\varphi(z) = \frac{kq}{a^2}\left[\sqrt{z^2 + a^2} - z\right]. \qquad (13)$$

Consider a time-dependent version of this problem, where the surface charge density is zero for $t < 0$, then somehow instantaneously becomes $q/\pi a^2$, uniformly distributed, for $t \geq 0$. Since there are no currents, **A** is zero and we have to calculate only $\varphi(z,t)$. For $t < 0$ there is no potential because there is no charge on the disc. After $t = 0$ there is charge on the disc, and that charge produces a nonzero field, but this change in the field propagates away from the disc at speed c. Thus the observer on the z axis detects zero potential until $t = z/c$. At that time the "first light" arrives at z from the center of the disc. For times $t \geq L/c$ where $L^2 = z^2 + a^2$ (Fig. 4), the potential reduces to its electrostatic value of Eq. (13). For times $z/c \leq t \leq L/c$, the part of the charged disc from which the news arrives at z of the sudden nonzero charge distribution, are points on a circle of radius $0 \leq \rho \leq a$. Points on this circle lie the distance r from the observation point, so that $t = r/c = (\rho^2 + z^2)^{1/2}$ or $\rho = (c^2t^2 - z^2)^{1/2}$. During this time interval, the a in Eq. (13) gets replaced with ρ, which gives $\varphi(z,t) = kq/(ct + z)$ for $z/c \leq t \leq L/c$. At $t = L/c$ Eq. (13) can be recovered by writing $\varphi(z,L/c) = kq\,(L - z)/(L^2 - z^2)^{1/2}$. **E** follows from Eq. (3).

In situation (B) the radiating point charge moves along a trajectory **r′**, a known function of time (Fig. 4). Since the source is a point charge, one may be tempted to write Eq. (7) as

$$\varphi(r,t) = \frac{kq}{|r - r'(t')|}. \qquad (14)$$

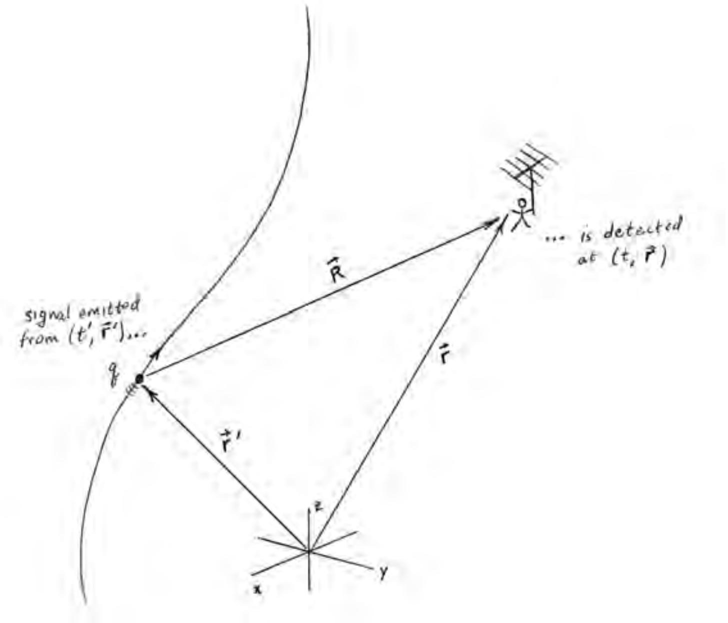

Fig. 4. The signals emitted by the charge at times t' are detected by the observer at time t.

But we must be careful here. Eq. (7) is formally an integral over a charge *density*. While we can accommodate the density of a point charge by using a Dirac delta function, a point charge is an idealization. The electrons (along with other leptons and the quarks) are the closest things we have to point charges in the real world, but they have nonzero magnetic dipole moments (neutrinos excepted), and how can there be such a thing as a point dipole? All we *know* about the electron is that if it has a finite size, that cannot exceed something like 10^{-18} m[10]—smaller than one meter by the same ratio as one second is to the age of the universe—but not known to be *exactly* zero! In addition, calculations with Feynman show we must take seriously the notion that an electron is always surrounded by a cloud of virtual electron-positron pairs. Even the humble electron carries a charge distribution! Whatever the charge distribution, let us smear the conceptual point charge out to a finite volume, then see what happens as we let the volume go to zero.

Once we entertain the notion of a very small but finite volume enclosing the charge, $|\mathbf{r} - \mathbf{r}'|$ is essentially constant within the volume, so instead of Eq. (14) we confront

$$\varphi(r,t) = \frac{k}{R}\int \rho(r',t')d^3r'. \qquad (15)$$

The integral describes a charge within a volume, but the charge is moving, and along with it so is the volume. That means Eq. (2) kicks in, and the volume containing the charge q is the volume V' of Eq. (2). Instead of Eq. (14) we obtain

$$\varphi(r,t) = \frac{kq}{s} \qquad (16)$$

where $\mathbf{R} = |\mathbf{r} - \mathbf{r}'(t')|$ and $s \equiv R - (\mathbf{R}\cdot\mathbf{v})/c$. When the same reasoning is applied the vector potential produced by a moving point charge, we find

$$A(r,t) = \frac{v}{c^2}\varphi(r,t). \qquad (17)$$

These are the Liénard-Wiechert potentials.[11] The factor $1/s$ that appears in $\varphi(\mathbf{r},t)$, instead of the $1/R$ that appears in the electrostatic potential, comes from the effects of the speed of a changing electromagnetic field being finite; it has nothing to do with the distribution of charge. Therefore the $1/s$ stays, even for a point charge.

Now by Eqs. (3) and (4) the **E** and **B** fields follow, again somewhat laboriously because of the ∇t_R terms. At the end of another day one finds for **E**,

$$E(r,t) = \frac{kq}{s^3}\left(1 - \frac{v^2}{c^2}\right)w + \frac{kq}{c^2 s^3}R\times(w\times a) \quad (18)$$

where **w** ≡ **R** – R(**v**/c) and **a** = d**v**/dt is the particle's acceleration. For **B** we obtain

$$B(r,t) = \frac{1}{c}\hat{R} \times E. \quad (19)$$

What about consistency between the Liénard-Wiechert potentials and the Jefimenko equations? If we apply the Jefimenko equations to a moving point charge, for the volumes in Eqs. (10) and (11) we must use Eq. (2). Richard Feynman wrote the results for **E** and **B** in a similar way, for their presentation to an introductory physics class. If the source is a point charge q moving with velocity **v**, the fields can be expressed as we find them in the *Feynman Lectures on Physics:*[12]

$$E(r,t) = kq\left[\frac{\hat{R}}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\hat{R}}{R^2}\right) + \frac{1}{c^2}\frac{d^2\hat{R}}{dt^2}\right]\Big|_{t-R/c} \quad (20)$$

with **B** given by Eq. (19). The last terms in all these expressions for **E** and **B**, the $1/R$ fields, detach themselves and propagate away from the sources. These fields, produced when charged particles accelerate, are the so-called radiation fields. They carry light and information between separated events.

A change in the electromagnetic field travels the fastest of any signal we know. Technology based on classical and quantum electrodynamics has indelibly changed human life. However, electrodynamics and the technologies it makes possible do not tell us what is worth communicating. Henry David Thoreau observed as much in 1854,[13] when electromagnetic state-of-the-art communication meant telegraph wires and Morse code. He wrote, "We are in great haste to construct a magnetic telegraph from Maine to Texas; but Maine and Texas, it may be, have nothing important to communicate."[13]. Today we can be constantly interrupted by text messages, phone calls and postings on smart phones, about which Thoreau again gently admonishes us, "If the bell rights, why should we run?"[14]

Be that is it may, the conceptual beauty of the electromagnetic fields coupled to matter, that make possible the exchange of information at a distance—and with it hopefully the growth of knowledge and wisdom and not mere distractions—may offer evidence that, unlike Thoreau's shepherd whose flocks wander to pastures higher than his thoughts,[15] our thoughts appreciate the strange and wonderful universe in which we are privileged to live. "Why should I feel lonely? is not our planet in the Milky Way?"[16 92]

Acknowledgments

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[6] First, the four first-order Maxwell equations are combined into two second-order inhomogeneous wave equations for the potentials. They are uncoupled by choosing the Lorenz gauge. Second, because the wave equation is linear, the potentials and sources are Fourier transformed from the time domain to the frequency domain, which reduces the wave equations to the time-independent Helmholtz equation to be solved a harmonic's amplitude in the potential. Third, the Helmholtz equation is by a second Fourier transform from the space to the wavenumber domain, and the potential's amplitude here is found by the method of Green's functions. These results and the reversal of the Fourier transforms give Eqs. (7) and (8). Fourth, this result is imported back into the inverse frequency-to-time domain Fourier inverse transform. Eqs. (5) and (6). See e.g., Wolfgang Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, Reading, MA, 1962), 240-245; J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, NY, 1975) 219-226. The intuitive guess method mentioned here is described and justified in David Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, Upper Saddle River, NJ, 1999) 422-425.

[7] Otto D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, NY, 1996), Sect. 15.7; Griffiths, Ref. 6, 427-429; Panofsky and Phillips, Ref. 6, 246-248 contains equivalent equations related to Jefimenko's by use of vector identities.
[8] Griffiths, Ref. 6, p. 430, offers a wonderful counterexample with acoustics, with a growling bear that can run at the speed of sound.
[9] E.g., Griffiths, Ref. 6, 444-454.
[10] So far no evidence for internal structure has been found for electrons up to the TeV energies. 1 TeV = $hf = hc/\lambda$ gives $\lambda \sim 10^{18}$ m.
[11] For an alternative, less "hand-waving" derivation of the Liénard-Wiechert potentials, consider the spacetime four-vector $\tilde{r} = (t, r)$ from which one constructs the four-vector from source point to observer point, $\tilde{R} = (t - t', r - r')$. Consider also the four-velocity of the source particle, $\tilde{u} = \gamma(1, v)$; and the four-potential $\tilde{A} = (\varphi, A)$. Consider these 4-vectors in the rest frame of the observer who sees the source particle move, and their counterparts in the rest frame of the source particle. By invoking the Lorentz-invariance of $\tilde{R}\cdot\tilde{A}$, $\tilde{u}\cdot\tilde{A}$, and $\tilde{u}\cdot\tilde{R}$, the resulting system of equations can be solved for the components of $\tilde{A}$, which are the Liénard-Wiechert potentials. See also Panofsky and Phillips, Ref. 6, 341-344; Griffiths, Ref. 6, 429-441; Jackson, Ref. 6, 654-693.
[12] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics, Vol. II* (Addison-Wesley, Reading, MA, 1964), 21-1.
[13] Henry David Thoreau, *Walden* (original 1854; referenced copy Houghton Mifflin Co., Boston, MA, 1960) 36.
[14] *ibid.,* 67.
[15] *ibid.,* 61.
[16] *ibid.,* 92.