Gravity as a Thin Lens

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The fall of 2015 marks the centennial of Albert Einstein's unveiling of the general theory of relativity. A dramatic result that Einstein derived was the prediction that a light ray grazing the Sun would be deflected about 1.7 seconds of arc. In 1919 a British expedition confirmed Einstein's prediction during a solar eclipse. The social significance of this event was as dramatic as its physics significance. A deep truth about the universe, predicted by a resident of Germany, was confirmed by a British group one year after World War I ended. On both sides this accomplishment lifted the public's imagination—as shown by Einstein suddenly becoming an international celebrity—to something higher than nationalistic squabbles.

In addition to the general relativity centennial, 2015 has also been designated by the United Nations as "The International Year of Light and Light-Based Technologies."[1] The organizers probably intended to emphasize recent technologies such as lasers and fiber optics and light-emitting diodes. But even more ubiquitous are lenses, which are easy to take for granted. Lenses, along with mirrors, were the first technologies for manipulating light. Lenses used for starting fires by focusing sunlight—so-called "burning lenses"—were cutting-edge technology a millennium ago. They were studied quantitatively in the tenth century by Muslim scholars, including Ibn al-Haytham (or Alhazen) and Abu Sàd al-Alá ibn Sahl; the latter's book, *On Burning Lenses and Mirrors* written in 984, describes refraction by glass with curved surfaces.[2] In the thirteenth century glassblowers in Venice and Pisa made magnifying glasses to aid monks in their reading. The first known image of eyeglasses held in a frame and poised on the nose of the reader appears in a fresco portrait of Cardinal Hugh of Saint-Cher, painted by Tommaso de Modena in 1352.

The simplest description of image formation by a lens is found in the "thin lens equation," a standard topic in introductory physics. For an object located at distance s_o from a thin lens of focal length f, the image forms at the distance s_i from the lens according to[3]

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}. \qquad (1)$$

The focal length, a property of the lens, is determined by the refractive index n of the glass and the first and second radii of curvature encountered by the ray, respectively R_1 and R_2. In terms of these parameters right-hand side of Eq. (1) is given by the "lens maker's equation,"

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \qquad (2)$$

A converging (diverging) lens has $f > 0$ ($f < 0$). Eqs. (1) and (2) can be derived through Fermat's Principle, which postulates that of all the possible paths connecting fixed points a and b, the path followed by a light ray is the one that makes the time of its trip a minimum:

$$\Delta t = \int_{t(a)}^{t(b)} dt = min. \qquad (3)$$

Since the speed of light in a refractive medium is $v = ds/dt = c/n$ with c the speed of light in vacuum, Eq. (3) can be rewritten

$$c\Delta t = \int_a^b n\, ds = min. \qquad (4)$$

For rays in the xy plane, $ds = [1 + (dy/dx)^2]^{1/2}\, dx$. The calculus of variations may be applied through the Euler-Lagrange equation to find the trajectory $y = y(x)$ that describes the ray, assuming n to be a known function of x and y.

The physics connection of interest here notes that, as Einstein and the British expedition together first showed, a massive body's gravity deflects light rays similar to the action of a converging lens. Thus we speak of "gravitational lensing." It might be fun to see how gravitational lensing can be written as a thin lens equation. To experts in gravitational lensing this is old stuff. However, those of us who do not think about gravitational lensing every day, but are familiar with thin lens physics along with some special relativity and intermediate mechanics, can recreate the connection for ourselves.

Let us start with some principles of special relativity, which applies in inertial reference frames and thus includes no gravitation. A laboratory frame measures locally the distance ds and time dt between two nearby events; and a rocket frame[4] measures locally the distance ds' and time dt' between those same two events. Even though in general $ds \neq ds'$ (recall length contraction) and $dt \neq dt'$ (viz., time dilation), nevertheless the *spacetime interval* is invariant, $(cdt)^2 - (ds)^2 = (cdt')^2 - (ds')^2$. The numerical value of this invariant is $(cd\tau)^2$ where $d\tau$ denotes the "proper time," the time between the two events as measured in the reference frame where they occur at the same place. If spatial coordinates are measured with a spherical coordinate system, with r the distance from the origin, θ the latitude measured from the north pole and φ the longitude, the spacetime interval finds expression as

$$d\tau^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \qquad (5)$$

In writing Eq. (5) I have absorbed c into the times, $cdt \to dt$ and likewise for proper time, so that time intervals are measured in meters. In addition I have omitted the plethora of parentheses

Image courtesy of SLAC National Accelerator Laboratory around the squares of the coordinate differentials. Furthermore, in the situations we will consider here, axial symmetry about the north-south (or z) axis will be assumed, so we may carry out our calculations in the $\theta = \pi/2$ plane, reducing Eq. (5) to

$$d\tau^2 = dt^2 - dr^2 - r^2 d\varphi^2 . \qquad (6)$$

Now let us "turn on gravity" by placing a static, spherically symmetric mass M at the origin. The gravitation of M will modify the spacetime interval of Eq. (6), because according to Einstein's principle of equivalence for gravitational and inertial mass, the presence of gravitation is equivalent to being in a non-inertial reference frame. Since M's gravity might stretch space in the radial direction, and since Einstein's field equations lead in general to the prediction that gravity affects a clock's period (viz.,, gravitational redshift), let us parameterize the effects of M by modifying Eq. (6) into

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\varphi^2, \qquad (7)$$

where the functions A and B are found by solving Einstein's gravitational field equations. This was done in 1915, with no approximation, by Karl Schwarzschild.[5] Schwarzschild found that $B = 1/A$ where (upon including c)

$$A(r) = 1 - \frac{2GM}{c^2}\frac{1}{r} \qquad (8)$$

and G denotes Newton's gravitation constant. Notice that $G/c^2 \approx 4.2\times10^{-28}$ m/kg, and thus $M^* \equiv GM/c^2$ is a length corresponding to mass M. For instance, if we insert the Sun's mass of 2×10^{30} kg, we find that $M^*_{sun} \approx 0.8$ km. When people say "the mass of the Sun is 0.8 kilometers" they mean $GM/c^2 = 0.8$ km. Henceforth we can write $A = 1 - 2M^*/r$. $2M^*$ is called the "Schwarzschild radius." The Schwarzschild radius of the Sun is 1.6 km; that of the Earth is about 0.44 cm. One can think of the Schwarzschild radius as the radius of the region within which M must be compressed to make it into a black hole. Here we will not be thinking about black holes *per se*, but are interested in any mass that (at least to first approximation) may be considered a point mass or spherically symmetric distribution of matter.[6]

Eq. (7) becomes (again absorbing the c's into the times)

$$d\tau^2 = A\,dt^2 - \frac{1}{A}dr^2 - r^2 d\varphi^2. \qquad (9)$$

Before going on, we should clarify what Schwarzschild coordinates mean.[7] Consider an imaginary spherical shell centered on and at rest relative to the origin, and enclosing the mass M. This shell's r-coordinate is by definition the shell's circumference divided by 2π. In the absence of gravitation this definition of r gives the same number as the distance from the origin to the shell. But gravity "stretches" space radially, making "radial length" and "r-coordinate difference" distinct quantities. In Schwarzschild geometry the radial distance as measured with a tape measure between Shell 1 and Shell 2 is *not* $|r_2 - r_1|$. The tape-measured distance dr_{shell} between nearby shells, and the difference dr in their r-coordinates, are related by $dr_{shell} = A^{-1/2}\,dr$. Therefore the distance Δr_{shell} between shells 1 and 2 is

$$\Delta r_{shell} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-\frac{2M^*}{r}}} . \qquad (10)$$

This integrates to a messy logarithm, but reduces to $r_2 - r_1$ as $M^* \to 0$. In traveling radially between Shell 1 to Shell 2, a traveler will go farther than $|r_2 - r_1|$. Similarly, when a clock at rest on a shell reads time interval dt_{shell} , this time measurement is related to the corresponding Schwarzschild time interval dt by $dt_{shell} = A^{1/2}\,dt$. Thanks to spherical symmetry, $d\varphi_{shell} = d\varphi$.

The Schwarzschild coordinates (t,r,φ) of an event must be inferred from measurements made *locally* by shell observers, who send their data to headquarters where it is transformed from their local values $(t_{shell}, r_{shell}, \varphi_{shell})$ into (t,r,φ). Such results are stitched together to make a global atlas in the Schwarzschild coordinates of events about M, and those coordinate differences fit together for pairs of nearby events according to Eq. (9).

From the interval of Eq. (9) we can work out the trajectory of a particle in free-fall about M.[8] We are interested in a beam of light, for which $d\tau = 0$. Furthermore, light can be considered to be a swarm of photons, each having zero mass. Before we find ourselves dividing by zero, let us work out the trajectory of a particle of nonzero mass m falling through a gravitational field. To describe light we will then take the limit as $d\tau \to 0$ and $m \to 0$.

Recall from gravity-free special relativity that in a given inertial frame the energy of a free particle of mass m is $E = mc^2\, dt/d\tau$ in conventional units with m measured kg or eV/c^2. Absorbing the c, this expression becomes $E = m(dt/d\tau)$ with m measured in electron volts. Also recall that, in Newtonian mechanics, for motion of a particle acted on by a central force, its angular momentum has magnitude $L = mr^2(d\varphi/dt)$. It turns out, in Schwarzschild spacetime, that E and L for a particle in free-fall are also constants of motion, and are given by

$$E = mA\frac{dt}{d\tau} \qquad (11)$$

and

$$L = mr^2\frac{d\varphi}{d\tau}. \qquad (12)$$

These expressions for E and L are derived in the Appendix. For now we see that they revert to the Special Relativity and Newtonian expressions for E and L in the appropriate limits. To proceed towards gravitational lensing, multiply Eq. (9) by $m^2A/d\tau^2$ then use Eqs. (11) and (12).[9] Thereby may Eq. (9) be re-arranged as

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - A\left[1 + \frac{L^2}{m^2}\frac{1}{r^2}\right]. \qquad (13)$$

To trace the trajectory of the light ray about the origin, we need to find r as a function of φ. Thanks to the chain rule and Eq. (12) we may swap $dr/d\tau$ for $dr/d\varphi$:

$$\frac{dr}{d\tau} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = \frac{dr}{d\varphi}\frac{L}{mr^2} \qquad (14)$$

which turns Eq. (13) into

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{E}{L}\right)^2 - A\left[\left(\frac{m}{L}\right)^2 + \frac{1}{r^2}\right]. \qquad (15)$$

With $d\tau$ gone and no m in any denominator, we can now set $d\tau = 0$ and $m = 0$ for light. It remains to determine E/L. Since L and E are constants of motion, their value anywhere along the trajectory will be the same as their values far from the m-M interaction region. Far away (at infinity) from M's gravity special relativity holds, so that L at infinity can be written in terms of m's linear momentum p as $L = L_\infty = p_\infty b = (E_\infty{}^2 - m^2)^{1/2}\, b$, where b denotes the impact parameter (Fig. 1), the lateral offset from a "bulls-eye" hit between m and M. Thus for a photon for which $m = 0$, $L/E = b$. These maneuvers turn Eq. (15) into

$$d\varphi = \pm \frac{dr/r^2}{\sqrt{\frac{1}{b^2} - \frac{A}{r^2}}}. \qquad (16)$$

Refer now to Fig. 2.

Fig. 1. Geometry of the impact parameter.

Let R be the r-coordinate for the point of closest approach (assumed to be outside the Schwarzschild radius) between the photon and M. Integrating Eq. (16) from $r = R$ to $r = \infty$ gives half the angle swept out by the line from the origin to the photon as the photon comes in from infinity, gets deflected by M, and moves out to infinity. Thus the total deflection $\Delta\varphi$ that follows from Eq. (16) will be

$$\Delta\varphi = 2\int_R^\infty \frac{dr/r^2}{\sqrt{\frac{1}{b^2} - \frac{A(r)}{r^2}}}. \qquad (17)$$

The impact parameter b can be written in terms of $A(R)$ as follows. With the help of $dr_{shell} = A^{-1/2}\, dr$ and $dt_{shell} = A^{1/2} dt$, it follows that $dr_{shell}/dt_{shell} = (1/A)dr/dt$. With $d\tau = 0$ in Eq. (9) for light, and with the help of Eqs (11) and (12) (the m will cancel out), and using $L/E = b$, it follows that

$$\left(\frac{dr_{shell}}{dt_{shell}}\right)^2 = 1 - \frac{A(r)b^2}{r^2}. \qquad (18)$$

At the point of closest approach $dr_{shell}/dt_{shell} = 0$, and Eq. (18) yields

$$b^2 = \frac{R^2}{A(R)}. \qquad (19)$$

Using Eq. (19) in Eq. (17), making the change of variable $u = R/r$, and with ample use of the binomial theorem (since $M^*/r << 1$), the integral becomes

$$\Delta\varphi \approx \pi + \frac{4M^*}{b}. \qquad (20)$$

If M were not present, $\Delta\varphi$ would equal π, so the *deflection* δ is $\Delta\varphi - \pi$, and thus

$$\delta = \frac{4M^*}{R}. \qquad (21)$$

We now have everything we need to treat the deflection of a light ray by a mass M as image formation by a thin lens (see Fig. 2). Consider a spherically symmetric distribution of mass M and radius R, and let a photon from a distant source approach M along a line that would result in grazing incidence. In the jargon of thin lens equations, let s_o denote the object distance—the distance between the photon source and M; and let s_i denote the image distance—the distance between M and the observer. The distances (or their r-coordinate correlates) s_o, s_i and R are illustrated in Fig. (2) which is *not* to scale, because $s_o >> R$ and $s_i >> R$. These are typically safe approximations-- for cosmological applications, the object and image distances could be hundreds of millions to billions of light-years, with R the size of a galaxy cluster, tens of millions of light-years. Closer to home, in the 1919 observations that affirmed Einstein, M^* and R are the Sun's mass and radius ($M^* \sim 1$km, $R \sim 10^{30}$ km), the image distance is eight light-minutes and the object distance (to another star) several light-years. In Fig. 2 we approximate the geometry as Euclidean.

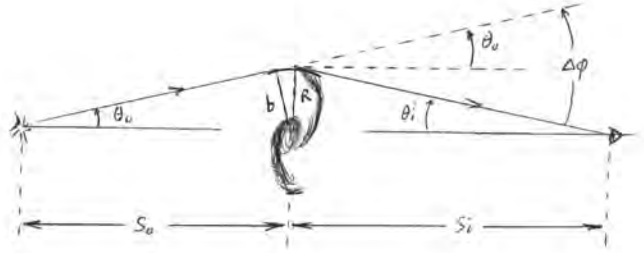

Fig. 2. Geometry for Eqs. (22) and (23).

In Fig. 2 we see that light leaves the object and grazes M's "surface" with impact parameter b, and that $\delta = \theta_o + \theta_i$ Since the angles are small, this can be written

$$\frac{4M^*}{R} = \frac{b}{s_o} + \frac{R}{s_i}. \qquad (22)$$

Recall that $M^* << R$ so that $A(R) \approx 1$ and thus $b \approx R$ is an approximation to Eq. (19), with which Eq. (22) becomes approximately

$$\frac{4M^*}{b^2} \approx \frac{1}{s_o} + \frac{1}{s_i}. \qquad (23)$$

This fits the template of a "thin lens equation," Eq. (1), where by Eq. (19) the focal length is given by the gravitational lensing version of the lens maker's equation,

$$\frac{1}{f} = \frac{4M^*}{R^2}\left(1 - \frac{2M^*}{R}\right) \qquad (24)$$

which to leading order in M^*/R is $1/f \approx 4M^*/R^2 > 0$, describing a converging lens. Comparing this focal length to that of an ordinary lens described by Eq. (2), it appears that, within the model and approximations made here, the mass distribution producing gravitational lensing behaves like a plano-convex lens whose curved surface has radius R and whose index of refraction is $n = 1 + 4M^*/R = 1 + \delta$.

We have produced from a chain of simple calculations an approximate equivalence between gravitational lensing and ordinary thin lenses. Of course, real gravitational lenses are more complicated than a sphere of radius R. But most refractive bodies are more complicated than lenses with perfectly spherical surfaces too. The surfaces of wide-angle, low-aberration lenses used in smart phone cameras are described by polynomials of eighth or tenth order or higher. Whether the application happens to be in optometry or astronomy, the thin-lens and lens maker's equations are first approximations to more realistic models. But this is the sort of thing that one does when doing physics: reduce a complicated phenomenon to something simple, even if it's not very accurate. That way one gains insight into the essentials, and gives more sophisticated models a special case to check against for consistency.

A popular illustration draws an analogy between the optical properties of the base of a wine glass and gravitational lensing, [10] When the gravitational lens matter lies precisely along the line between the observer and a distant galaxy being imaged, the image seen is a ring, the so-called Einstein ring. Similarly, when looking at a candle through the base of a wine glass, if the center of the base sits precisely between the candle flame and the observer (and the stem coincident with the line of sight), a full ring is seen. If the wine glass base or the gravitational lens are not perfectly aligned between the object and the observer's eye, then skewed images perpendicular to the ring are seen.

Raise your glass to Einstein's rings, gravitational lensing, to ordinary lenses, and to the International Year of Light!

Appendix: Energy and Angular Momentum of a Particle in Free-Fall in Schwarzschild Spacetime

Fermat's principle for geometrical optics asserts that, of all paths that connect two fixed points *a* and *b* in space, a light ray that goes between those points follows the path for which the elapsed time is a minimum. A "Fermat's Principle" exists for particles in free-fall as described by general relativity. It says that of all trajectories connecting two fixed events *a* and *b* in spacetime, the trajectory actually followed by a particle in free-fall is the one for which the elapsed proper time is a maximum, $\int_a^b d\tau = max.$ [11] When spacetime is described by Schwarzschild coordinates, upon factoring out the *dt* from the right-hand side of (7) to give $d\tau = \Lambda dt$ where $\Lambda = \sqrt{A(r) - \frac{\dot{r}^2}{A(r)} - r^2\dot{\varphi}^2}$, and the overdots denote derivatives with respect to *t*, the "Fermat's principle" for free-fall says

$$\int_{t(a)}^{t(b)} \sqrt{A - \frac{\dot{r}^2}{A} - r^2\dot{\varphi}^2}\; dt = max. \qquad (26)$$

In the language of the calculus of variations, Λ is recognized as the Lagrangian. There are two canonical momenta, $p_r = \frac{\partial\Lambda}{\partial\dot{r}} = -\frac{\dot{r}}{A\Lambda}$ and $p_\varphi = \frac{\partial\Lambda}{\partial\dot{\varphi}} = -\frac{r^2\dot{\varphi}}{\Lambda}$. The Hamiltonian is defined according to $H = p_r\dot{r} + p_\varphi\dot{\varphi} - \Lambda$, which gives $H = -A/\Lambda$. The equations of motion (Euler-Lagrange equations) yield two conservation laws: since $\partial\Lambda/\partial\varphi = 0$, the momentum p_φ conjugate to φ is conserved, so that $r^2(d\varphi/d\tau) = const.$, whose Newtonian limit ($d\tau \to dt$) will be recognized as angular momentum per mass, *L/m*. Another version of the Euler-Lagrange equation says that $-\frac{\partial\Lambda}{\partial t} = \dot{H}$, and because $\partial\Lambda/\partial t$ vanishes, $H = -A/\Lambda = const.$ But $d\tau = \Lambda dt$, and thus $H = -A(dt/d\tau) = const.$ which we identify as $-E$ with *E* the mechanical energy, because in non-dissipative mechanics, conservation of the Hamiltonian means conservation of energy, and the expression for a free particle's energy in special relativity is $E/m = dt/d\tau$ and $A \to 1$. Thus in Schwarzschild geometry we identify $A(dt/d\tau)$ with E/m.

For an alternate derivation of *E* and *L* that does not make explicit use of the Euler-Lagrange equations, canonical momenta and the Hamiltonian, see Taylor and Wheeler, Ref. 7, Chs. 3 and 4.

Acknowledgments

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Notes and References

[1] UN Year of Light and Light-Based Technologies, http://www.light2015.org/Home.html.

[2] *Elegant Connections in Physics: Light as Nexus in Physics*, *Radiations*, Fall 2014, 20-25 and references therein, including R. Rashed, "Ibn Sahl on burning mirrors and lenses," *Isis* **81**, 464-491 (1990).

[3] See the geometrical optics chapters of any introductory physics text, or *Elegant Connections in Physics* "Foundations of Geometrical Optics: Phenomenology and Principles," *SPS Observer* (Summer 2010), (http://www.spsobserver.org). In geometry and kinematics "distance" and "length" are non-negative quantities, but in the thin lens equation they may have either sign.

[4] The mental pictures of "lab frame" and "rocket frame" are borrowed from Edwin Taylor and John A. Wheeler's *Spacetime Physics* (W.H. Freeman, 1966, 1993).

[5] For the steps in the solution to Einstein's field equations that give the Schwarzschild solution, see any general relativity text, e.g., Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley & Sons., New York, NY, p. 179). Because the general relativity equations are non-linear, Einstein was impressed that someone had been able to find an exact solution. In the applications of general relativity he worked out, such as the anomalous precession of Mercury, Einstein used perturbation theory.

[6] Proof of the point-sphere equivalence in general relativity is called Birkhoff's theorem.

[7] Edwin Taylor and John A. Wheeler, *Exploring Black Holes: Introduction to General Relativity* (Addison Wesley Longman, San Francisco, 2000), Ch. 2.

[8] One could also consider a freely falling observer, which would introduce a relative velocity between the frames. Here we consider a shell observer at rest on a shell with coordinate *r,* and the headquarters Schwarzschild-coordinate observer; both are at rest relative to *M*.

[9] See also Taylor and Wheeler, Ref. 7, Chs 4 and 5.

[10] Treu Tommaso, "Strong Lensing by Galaxies, *Annual Review of Astronomy and Astrophysics* **48** (2010), 87-125; Curtis McCully, "New Insights Into Peculiar Thermonuclear Supernovae and Line of Sight Effects in Gravitational Lensing," PhD dissertation, Rutgers University, New Brunswick, NJ (2014), 15-22; for a visual demonstration with the wine glass see Phil Marshal, "What is Gravitational Lensing?" (SLAC video), https://www.youtube.com/watch?v=PviYbX7cUUg;

[11] For a connection between the "Fermat's principle of general relativity" and Hamilton's principle, see D.E. Neuenschwander. *Emmy Noether's Elegant Theorem* (Johns Hopkins University Press, Baltimore, MD, 2011), Sec. 3.5.