Coupled Oscillations in Diverse Phenomena

Part 1: A simple mathematical model

by Dwight E. Neuenschwander, Southern Nazarene University

If you are a musician or a music appreciator, you have encountered "sympathetic vibrations," where striking one string of a piano or guitar makes nearby unstruck strings vibrate. As the vibrations of the struck string vibrate the surrounding air, the jiggling air makes neighboring strings vibrate too. Sympathetic vibrations are examples of *coupled oscillations.* A simple mathematical model of a coupled oscillator also beautifully describes diverse phenomena from ammonia masers to neutrino oscillations. The parameters are different, but they are described by essentially the same equations.

Coupled oscillations are manifestly visible in a classroom demonstration with two pendulums whose upper ends are attached to a common rod. When pendulum 1 swings it communicates its motion to the coupling rod, which eventually sets pendulum 2 in motion. Due to conservation of energy, as one pendulum swings more, the other swings less. As long as damping is negligible, the two pendulums take turns swinging with the larger amplitude.

To make a simple mathematical model that gets at the essence of the coupling, begin with two springs. Let the left and right spring have spring constant k, each attached to identical masses m. The springs are coupled by connecting them with a third spring with a spring constant k' (Fig. 1).

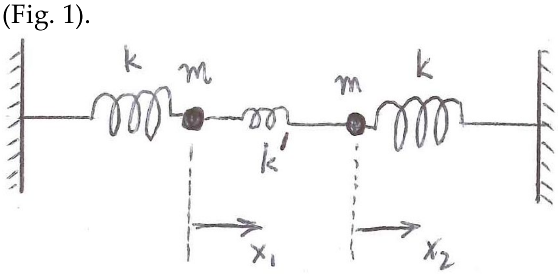

Figure 1. A coupled oscillator represented with springs with spring constants k, k', and k.

Apply Newton's second law to each of the two masses, using Hooke's law to model the forces exerted on them by the springs. With the coordinates of Fig. 1, for the left spring (spring 1) we have

$$-kx_1 - k'(x_1 - x_2) = m\ddot{x}_1 \qquad (1a)$$

and for the right spring (spring 2) Newton's second law gives

$$-kx_2 - k'(x_2 - x_1) = m\ddot{x}_2 \qquad (1b)$$

where overdots denote time derivatives. Regroup Eqs. (1) to collect coefficients of each coordinate:

$$-(k + k')x_1 + k'x_2 = m\ddot{x}_1 \qquad (2a)$$

$$k'x_1 - (k + k')x_2 = m\ddot{x}_2. \qquad (2b)$$

Equations (2) present us with two equations and two unknowns for $x_1(t)$ and $x_2(t)$. We can try two strategies: (a) solve for $x_1(t)$ and $x_2(t)$ separately, or (b) think of the two masses and the three springs as a single system and find the modes of its oscillation where each mode has a unique frequency—the so-called "normal modes," a.k.a. the "eigenstates" of the system, with their corresponding frequency "eigenvalues" (German *eigen* = English *own*). Let's look first at strategy (a). Then we'll turn to strategy (b).

(a) Solving for $x_1(t)$ and $x_2(t)$

To solve for $x_1(t)$ and $x_2(t)$ separately, since Eqs. (2) are linear in both variables they can be easily combined by superposition into two other equations, one for $x_1(t) + x_2(t) \equiv \sigma(t)$ and another one for $x_1(t) - x_2(t) \equiv \delta(t)$. Adding Eqs. (2a) and (2b) gives

$$\ddot{\sigma} + \omega_\ell^2 \sigma = 0 \quad (3a)$$

whereas subtracting Eq. (2b) from (2a) produces

$$\ddot{\delta} + \omega_h^2 \delta = 0 \quad (3b)$$

where

$$\omega_\ell \equiv \sqrt{\frac{k}{m}} \qquad (4a)$$

is a lower frequency and

$$\omega_h \equiv \sqrt{\frac{k+2k'}{m}} \qquad (4b)$$

is a higher frequency. Equations (3) give the solutions

$$\sigma(t) = a\sin(\omega_\ell t) + b\cos(\omega_\ell t) \quad (5a)$$

$$\delta(t) = p\sin(\omega_h t) + q\cos(\omega_h t) \quad (5b)$$

where a, b, p, and q are constants. Suppose the initial conditions include $\dot{x}_1(0) = \dot{x}_2(0) = 0$, with both masses initially at rest. This means $\dot{\sigma}(0) = 0 = \dot{\delta}(0)$, which gives $a = p = 0$. But to produce oscillations, one of them has been displaced from equilibrium, then released at $t = 0$. Therefore, let $x_1(0) = b$ and $x_2(0) = 0$, which implies $q = b$. Now Eqs. (5) are

$$\sigma(t) = b\cos(\omega_\ell t) = x_1(t) + x_2(t) \qquad (6a)$$

and

$$\delta(t) = b\cos(\omega_h t) = x_1(t) - x_2(t). \qquad (6b)$$

Equations (6) can be inverted to give

$$x_1(t) = \frac{b}{2}[\cos(\omega_\ell t) + \cos(\omega_h t)] \qquad (7a)$$

and

$$x_2(t) = \frac{b}{2}[\cos(\omega_\ell t) - \cos(\omega_h t)]. \qquad (7b)$$

The trig identities

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad (8a)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \quad (8b)$$

allow Eqs. (7) to be written in a way that exhibits beats. The two frequencies result in a rapid oscillation of frequency $(\omega_h + \omega_\ell)/2$, which is modulated by a slowly oscillating envelope of frequency $(\omega_h - \omega_\ell)/2$.[1]

$$x_1(t) = 2b\cos\left(\frac{(\omega_\ell+\omega_h)t}{2}\right)\cos\left(\frac{(\omega_h-\omega_\ell)t}{2}\right) \quad (9a)$$

$$x_2(t) = 2b\sin\left(\frac{(\omega_\ell+\omega_h)t}{2}\right)\sin\left(\frac{(\omega_h-\omega_\ell)t}{2}\right) \quad (9b)$$

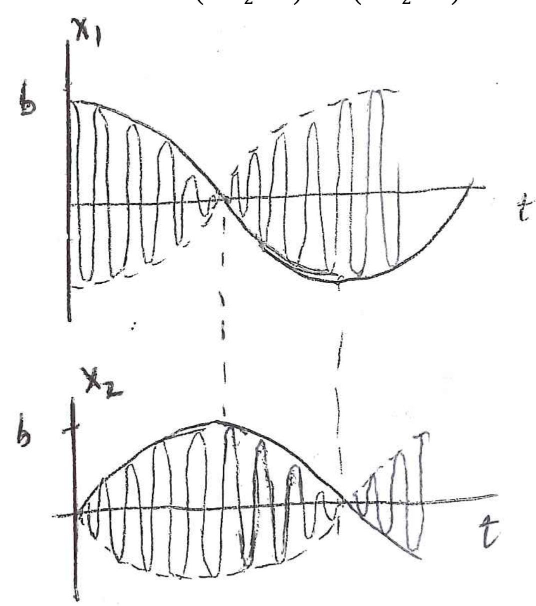

Figure 2. An illustration of the motions $x_1(t)$ & $x_2(t)$.

(b) The Normal Modes

Return to Eqs. (2) and arrange them into a matrix equation,

$$\begin{pmatrix} -(k+k') & k' \\ k' & -(k+k') \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}. \quad (10)$$

Introducing column vectors and deploying the Dirac bracket notation,[2] let

$$|x\rangle \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad (11)$$

Define a matrix K of spring constants:

$$K \equiv \begin{pmatrix} (k+k') & -k' \\ -k' & (k+k') \end{pmatrix}. \qquad (12)$$

With Eqs. (11) and (12), Eq. (10) takes the abbreviated form

$$-K|x\rangle = m\frac{d^2|x\rangle}{dt^2}. \qquad (13)$$

In strategy (a) the two-mass system, as mapped by $x_1(t)$ and $x_2(t)$, had two oscillations with different frequencies going on at the same time. We now set for ourselves the task of finding single-frequency modes of oscillation of the entire two-mass, three-spring system. This means we seek a set of two new position variables where each oscillates with some as-yet-unknown constant angular frequency. Let's call these two modes $\eta_1 \equiv A_1\cos(\omega t)$ and $\eta_2 \equiv A_2\cos(\omega t)$, where ω is a constant angular frequency (or set of constant frequencies) to be determined. Let $|\eta(t)\rangle$ denote these coordinates arranged as a special case of generic states $|x\rangle$,

$$|\eta(t)\rangle = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\cos(\omega t) \equiv |A\rangle\cos(\omega t) \qquad (14)$$

where $|A\rangle$ is time-independent. Insert these coordinates—this eigenvector—into Eq. (13), which becomes (after cancelling $\cos(\omega t)$)

$$K|A\rangle = m\omega^2|A\rangle. \qquad (15a)$$

Transpose this into the form

$$\begin{pmatrix} -m\omega^2+(k+k') & -k' \\ -k' & -m\omega^2+(k+k') \end{pmatrix}|A\rangle = |0\rangle \qquad (15b)$$

where $|0\rangle$ means the zero vector. Now we appeal to the theorem of alternatives, a.k.a. the invertible matrix theorem.[3] Recall what it says about a homogeneous matrix equation of the form

$$M|x\rangle = |0\rangle \qquad (16)$$

with square matrix M and column matrix $|x\rangle$; Eq. (15b) is an example of such a matrix equation. There are two alternatives for $|M|$, the determinant of M: it is either zero or it is nonzero. If $|M| \neq 0$ the theorem shows that Eq. (16) has the unique but trivial solution $|x\rangle = |0\rangle$. But if $|M| = 0$ the theorem guarantees a nontrivial but not unique solution. Since we want a nontrivial solution, we set to zero the determinant of the square matrix in Eq. (15b) and solve for ω. We get

$$\omega = \sqrt{\frac{k+k'\pm k'}{m}} \qquad (17a)$$

choosing the positive sign when taking the square root since frequencies are non-negative. There are two distinct roots. The minus sign under the radical gives

$$\omega_\ell = \sqrt{\frac{k}{m}} \qquad (17b)$$

and from the plus sign,

$$\omega_h = \sqrt{\frac{k+2k'}{m}} \qquad (17c)$$

[compare to Eqs. 4]. These results—two angular frequency eigenvalues—means there are two modes of vibration, each with its distinctive frequency. Let's see what kinds of vibrations these eigenvalues imply. To do this we take one eigenvalue at a time, insert it into Eq. (15a), and see what ratio A_2/A_1 results. without further assumptions we can't do better than the *relative* values of these amplitudes because the theorem of alternatives promised a nontrivial solution but not a unique one.

Begin with ω_ℓ of Eq. (17b). When inserted in Eq. (15a) this gives the pair of equations

$$k'A_1 - k'A_2 = 0 \qquad (18a)$$

$$-k'A_1 + k'A_2 = 0. \qquad (18b)$$

Both yield $A_1 = A_2 \equiv a.$ Label as $|\eta_\ell\rangle$ the state $|\eta\rangle$ that has ω_ℓ for its eigenvalue. Omitting the time dependence in the eigenstates for now by setting $t = 0$ in Eq. (14), so far we have

$$|\eta_\ell(0)\rangle = a\begin{pmatrix}1\\1\end{pmatrix}. \qquad (19a)$$

When we repeat this procedure but insert ω_h into Eq. (15a), we find that $A_2 = -A_1$ so that at $t = 0$ we find this eigenstate $|\eta_h\rangle$ to be

$$|\eta_h(0)\rangle = a\begin{pmatrix}1\\-1\end{pmatrix}. \qquad (19b)$$

Restoring the time dependence via Eq. (14), within the scaling factor a we now have the eigenstates with their respective eigenvalues:

$$|\eta_\ell(0)\rangle = a\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_\ell t) \qquad (20a)$$

$$|\eta_h(0)\rangle = a\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_h t). \qquad (20b)$$

What kinds of motions do these states describe? Let us appropriate Eq. (11) for each of the eigenvectors. For $|\eta_\ell(t)\rangle$ we can write

$$|\eta_\ell(0)\rangle = a\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_\ell t) = \begin{pmatrix}x_1\\x_2\end{pmatrix} \qquad (21a)$$

which says that $x_1 = x_2$. If spring 1 moves its mass to the right by 1 cm, then spring 2 simultaneously moves its mass to the right by 1 cm too. This behavior is illustrated in Fig. 3a. For the eigenvector $|\eta_h(t)\rangle$ we have

$$|\eta_h(0)\rangle = a\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_h t) = \begin{pmatrix}x_1\\x_2\end{pmatrix} \qquad (21b)$$

which says that $x_1 = -x_2$. If spring 1 moves its mass to the right by 1 cm, then spring 2 simultaneously moves its mass to the left by 1 cm. This behavior for the coupled oscillators is illustrated in Fig. 3b. When vibrating in one of the eigenstate modes, the two masses move together in a single-frequency choreography.

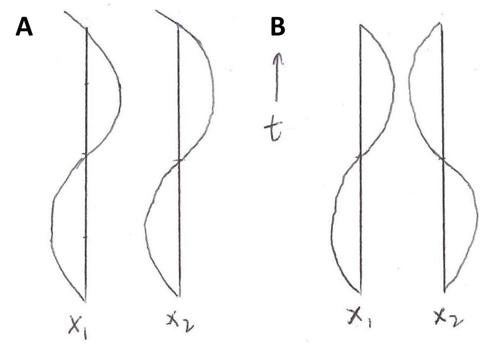

Figure 3. The two eigenstate modes for the coupled oscillator: (a) the mode of Eq. (21a) with angular frequency ω_ℓ, and (b) the mode of Eq. (21b) of angular frequency ω_h.

Eigenvectors as a Basis

The original set of state vectors of Eq. (11) expresses the instantaneous state of the coupled oscillators. Analogous to how a vector **r** in the Euclidean plane can be written as $\mathbf{r} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}}$, the abstract state vector $|x\rangle$ can be split into a superposition of "basis vectors" as follows:

$$|x\rangle = \begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix} \equiv x_1|1\rangle + x_2|2\rangle \qquad (22)$$

where

$$|1\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \qquad (23a)$$

and

$$|2\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}. \qquad (23b)$$

These are the basis vectors of the "1-2 basis." Let $\langle x|$ be the row vector made by transposing the column vector $|x\rangle$ (when we get to quantum mechanics, which uses complex numbers, $\langle x|$ will be the transpose and complex conjugate of $|x\rangle$). Then by the rules of matrix multiplication, the scalar product (a.k.a. the "dot product") between two vectors can be written $\langle x|y\rangle$. Like $\mathbf{\hat{i}}$ and $\mathbf{\hat{j}}$, we see that $|1\rangle$ and $|2\rangle$ are also "orthonormal", a hybrid word which means they are orthogonal: $\langle 1|2\rangle = 0$; and they are unit vectors, i.e., "normalized" to have magnitude unity: $\langle 1|1\rangle = \langle 2|2\rangle = 1$. An orthonormality condition can be summarized by saying $\langle i|j\rangle = \delta_{ij}$, where δ_{ij} denotes the "Kronecker delta," equal to 1 if $i = j$ and 0 if $i \neq j$. Any state vector in two-dimensional space can be written as a superposition of $|1\rangle$ and $|2\rangle$, as Eq. (22) illustrates. As basis vectors, $|1\rangle$ and $|2\rangle$ are said to "span the space." Equivalent to the statement that $|1\rangle$ and $|2\rangle$ form an orthonormal basis is the statement that they satisfy the "completeness relation,"[2]

$$|1\rangle\langle 1| + |2\rangle\langle 2| = \tilde{1} \tag{24}$$

where $\tilde{1}$ denotes the unit matrix, which in the case before us is 2×2. *The eigenstates form another orthonormal basis in the same space*.

We have seen how the theorem of alternatives guarantees nontrivial but nonunique solutions. In our example this shows up in the arbitrary scale factor *a* that, in this system, is common to $|\eta_\ell(t)\rangle$ and $|\eta_h(t)\rangle$. To determine *a* we push farther. We have the freedom to make these eigenvectors be unit vectors. In so doing they form a set of orthonormal basis vectors.[4] Eigenvectors do not have to be *unit* vectors in order to be a basis, but why would you give $\mathbf{\hat{i}}$ or $\mathbf{\hat{j}}$ a magnitude of 2 or 17—those factors would have to be divided out of your calculations. So let's normalize the eigenvectors $|\eta_\ell(t)\rangle$ and $|\eta_h(t)\rangle$. This determines the scale factor *a*. Requiring $\langle\eta_\ell|\eta_\ell\rangle = 1$ and $\langle\eta_h|\eta_h\rangle = 1$ gives $a = 1/\sqrt{2}$ in both cases, and therefore from Eqs. (19),

$$|\eta_\ell(0)\rangle. = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{25a}$$

and

$$|\eta_h(0)\rangle. = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{25b}$$

You can verify their orthogonality, $\langle\eta_\ell|\eta_h\rangle = 0$, and, now normalized, that they satisfy the completeness relation at *t* = 0,

$$|\eta_\ell(0)\rangle\langle\eta_\ell(0)| + |\eta_h(0)\rangle\langle\eta_h(0)| = \tilde{1}. \tag{26b}$$

The next installment will investigate the connection to two-state systems and neutrino oscillations.

References

[1] If you are used to "the beat frequency" being $|\omega_h - \omega_\ell|$ as encountered in musical acoustics, note that the frequency of the *intensity* (what we hear) is proportional to the square of the wave function, and thereby the intensity has twice the frequency of the wave function.

[2] D.N., "Synthesis & Analysis, Part 1: The Completeness Relation and Dirac Brackets," *SPS Newsletter*, Oct. 1996, 10-12.

[3] David Lay, *Linear Algebra and its Applications* 3rd. ed. (Boston, MA: Pearson Addison-Wesley, 2006), 128-130.

[4] Any vector can be normalized to have unit length simply by dividing the vector by its magnitude. But if the eigenvectors are not already orthogonal, which happens when two eigenfunctions share the same eigenvalue, one can make linear combinations of them to form an orthogonal set. The systematic way of doing so is called the "Schmidt orthogonalization procedure." See Eugen Merzbacher, *Quantum Mechanics,* 2nd Ed. (New York: Wiley, 1970), 149, 317-318.